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Linear relationship statistics in diffusion limited aggregation

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Abstract

We show that various surface parameters in two-dimensional diffusion limited aggregation (DLA) grow linearly with the number of particles. We find the ratio of the average length of the perimeter and the accessible perimeter of a DLA cluster together with its external perimeters to the cluster size, and define a microscopic schematic procedure for attachment of an incident new particle to the cluster. We measure the fractal dimension of the red sites (i.e., the sites such that cutting each of them splits the cluster) as equal to that of the DLA cluster. It is also shown that the average number of dead sites and the average number of red sites have linear relationships with the cluster size.

(Some figures in this article are in colour only in the electronic version)

Diffusion limited aggregation (DLA) is a model of a growing cluster, originally proposed by Witten and Sander [1]. The model has been shown to underlie many pattern forming processes including dielectric breakdown [2], electrochemical deposition [3], viscous fingering and Laplacian flow [4] etc. It is defined by a simple stochastic model on a square lattice as follows. A *seed* particle is located at the center of the lattice, and then a random walker is released from infinity—operationally, from a point at a radial distance much larger than the radius of the growing cluster. Upon contacting, the random walker sticks irreversibly to the cluster. Repeating the process leads to an intricate and ramified structure whose surface in the plane grows proportionally to the bulk (this will be shown in this paper).

This procedure is equivalent to solving Laplace's equation outside the aggregated cluster with appropriate boundary conditions. In two dimensions, since analytic functions automatically obey Laplace's equation, the theory of conformal mappings provides another mechanism for producing the shapes. This method has been directly used by Hastings and Levitov to study DLA [5].

One of the most interesting aspects of such an aggregate is the multifractal behavior of the growth site probability distribution (the harmonic measure) $\{p_i\}$, where p_i is the probability that the site i, belonging to the perimeter of the cluster, will grow at the next time [6–8]. The screened sites with tiny growth probability play an important role in determining the multifractality, while evaluating the harmonic measure on these sites is too difficult. The theoretical difficulty emerges from solving equations with boundary conditions on a complicated growing interface.

Different numerical methods applied for large aggregates in a plane show that DLA does not follow a simple fractal pattern and deviates from linear self-similarity [9]. Although most of the studies have been focused on the scaling behavior of different quantities in DLA, here in this paper we show that there exist a couple of linear relations which, to our knowledge, have not been addressed yet. We show that the perimeter and the accessible perimeter of a growing aggregate, together with their external perimeters, grow linearly with the cluster size n, with different rates. The number of sites which are not accessible to the incident random walker will be shown to be increased linearly with the size. Our results indicate that for most particle attachments (in which a particle is assumed to be at the center of a square of the mesh size a) the choice is *single-side* contact. We also use the concept of *red sites* as a measure of ramification of the aggregates and compute their fractal dimension. The average number of red sites is another quantity which has a linear relationship with the number of aggregated particles.

In order to investigate the behavior of the aforementioned quantities, we simulated several independent on-lattice DLAs of different mass up to $n = 10^5$ particles. One can see that an ensemble of simulated DLAs is strongly fluctuating, and to

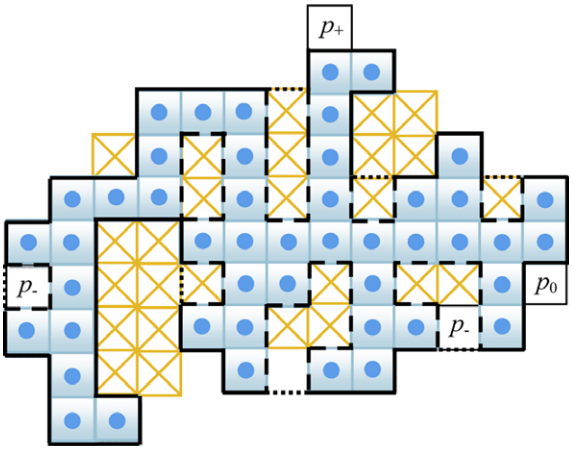

Figure 1. A putative small DLA cluster of n aggregated particles shown in shaded squares. The squares marked with $\times$ show the dead sites which are inaccessible to the random walker coming from infinity. The hull of the DLA, defined in the text, is the union of the solid and dashed lines. The outer perimeter is the union of the solid and dotted lines. The accessible perimeter is the hull of the union of the DLA cluster and the dead sites. The $(n+1)$th incident random walker may make a positive, negative or zero contribution to the length of the hull, denoted by p_+, p_- and p_0, respectively.

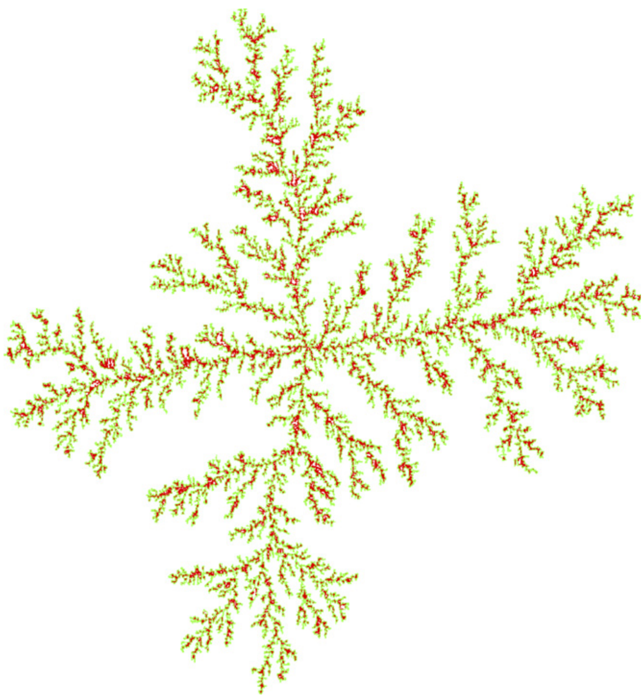

Figure 2. The hull (dark) and the accessible perimeter (light) of a DLA cluster of size $n = 10^5$.

obtain consistent results the averages have to be taken over a large number of samples. In this paper, the averages are taken over 5000 independent samples for different cluster sizes.

The first object that we investigate is the perimeter or *hull* of an aggregate. Consider a cluster of aggregated particles on a square lattice (see figure 1), where each frozen particle in the cluster is assumed to be located at the center of a plaquette. To define the hull, a walker moves clockwise around the aggregate and along the edges of the corresponding lattice starting from a given boundary edge on the cluster. The direction at each step is always chosen such that walking on the selected edge leaves a frozen particle on the *right* and an empty plaquette on the *left* of the walker. If there are two possible ways of proceeding, the preferred direction is that to the right of the walker. The directions *right* and *left* are defined locally according to the orientation of the walker. This algorithm yields the hull of the DLA whose length l (in units of the mesh size a) is equal to the number of steps until the walker returns to the starting edge.

Let us denote the length of the hull of a cluster of size n by l_n. We now show that l_n must be proportional to the cluster size.

To have a cluster of size $n+1$, a random walker is released from infinity. It sticks to an edge of the hull with a probability proportional to the harmonic measure there. We define the three possibilities as shown in figure 1, during which the random walker can stick to the cluster having *single-side*, *double-side* or *triple-side* contacts. Upon selecting each of them, it may make a positive, negative or zero contribution to the length of the updated hull. According to its contribution, we denote the probability that the random walker selects each of the three possibilities as p_+, p_- and p_0, respectively. It can be easily checked that the events with single-side contacts always make a positive contribution of $+2$ to the length of the hull, while the two other possible events can make either zero contribution or a negative contribution of $2n_-$, with $n_- \geqslant 1$. One can indeed define the length of the updated hull by considering these contributions to the previous length l_n, using the following recursive relation:

$$l_{n+1} = l_n + 2[p_+ - n_- p_-]. \qquad (1)$$

As our following experiments suggest, for $n \gg 1$, that the last term in the above relation seems to be independent of the cluster size and, therefore, one can obtain that $l_n \sim 2[p_+ - n_- p_-]n$. Our simulation result for the average length of the hull l_n as a function of the cluster size n is in good agreement with this linear relationship; see figure 3. We find that $\frac{l_n}{n} = 1.791(2)$. This yields the infimum of the probability that the incident random walker chooses a *single-side* contact, and hence we obtain $p_+ \gtrsim 90\%$.

In addition to studying the hull of the aggregates, we also study the external perimeter of the hull which can measure the number of sites that get trapped in the fjords (proportional to n_-). To define the external perimeter, we first close off all the narrow passageways of a lattice spacing on the DLA cluster and then look at the hull of the resulting cluster (see figure 1). We find experimentally that the length of the external perimeter l'_n also has a linear relationship with the cluster size n. The best fit to our data gives $\frac{l'_n}{n} = 1.325(2)$.

All sites on the hitherto considered perimeters are not necessarily accessible to the incident random walker in all regions of the aggregate. Therefore, it is of interest to measure the totally accessible perimeter which is, in principle, a *hull* that surrounds the union of the DLA cluster and all sites inaccessible to the incident random walker coming from infinity—see figures 1 and 2.

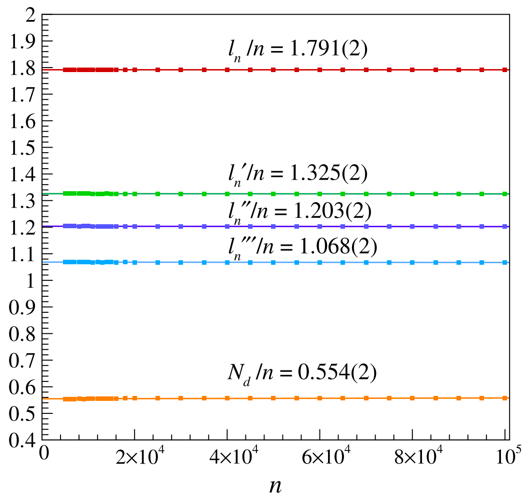

Figure 3. From top to bottom: the rescaled length of the perimeter l_n, external perimeter l_n', accessible perimeter l_n'', accessible external perimeter l_n''', and rescaled number of dead sites N_{d} versus the cluster size n. The errors are less than the symbol size. The solid lines indicate the best linear fit of zero slope.

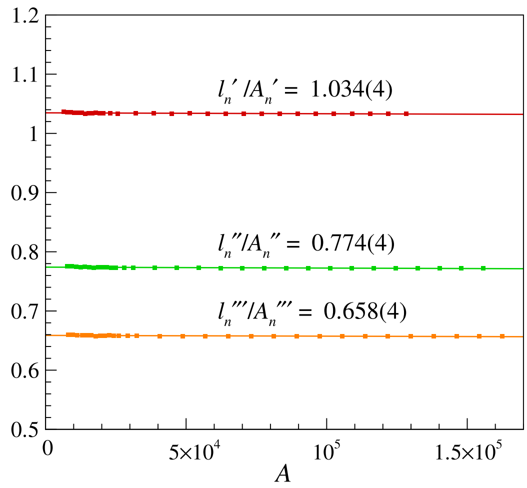

Figure 4. Linear relationships between different perimeter lengths and the area enclosed by them. From top to bottom: the length of the external perimeter l_n', accessible perimeter l_n'' and accessible external perimeter l_n''', rescaled by their enclosed areas denoted by A_n', A_n'' and A_n''', respectively.

In order to determine the accessible perimeter, we introduce an algorithm which seems to be more efficient than one used in [11]. This may be called the '*burning algorithm*', during which each accessible site around the cluster will be marked as a *burning* site, and all cluster sites and inaccessible ones will be left unmarked.

The algorithm begins by drawing a box that includes the entire cluster without touching it, and 'marking in' from the boundary of the box. A boundary site of the box is selected and is marked as '*free*' if there is no cluster site in its nearest neighborhood and as '*burnt*' otherwise. If the site is left 'free' marked, after checking all of its nearest neighbor sites and marking each of them as 'free' or 'burnt' as before, it is recolored as 'burnt'. Repeating this procedure for all sites that are 'free' marked and all unmarked boundary sites of the box, the inside of the box will be partitioned into two regions; one contains the sites which are being marked as 'burnt' sites and all of which are accessible to the random walker, and the other region, i.e., the inaccessible region, is the union of the cluster sites and unmarked sites (or *dead* sites) which are not accessible to the random walker. The accessible perimeter is then the hull of the inaccessible region.

We find that the average length of the accessible perimeter l_n'' and the average number of dead sites N_{d} grow linearly with the cluster size n. The linear relations can be obtained as shown in figure 3, according to $\frac{l_n''}{n} = 1.203(2)$ and $\frac{N_{\mathrm{d}}}{n} = 0.554(2)$, respectively.

Following the same ratiocination as for equation (1) for these observations, one can estimate the probability that the incident particle sticks to the accessible perimeter of the DLA with a single contact. We denote the same probabilities as before, that the incident random walker coming from infinity sticks to the accessible perimeter with *single-side*, *double-side* or *triple-side* contacts, by p_+', p_-' and p_0', respectively. Note that for the hull of the accessible perimeter, all *single-side*, *double-side* and *triple-side* contacts always make positive, zero and negative contributions (of $+2$, 0 and -2, i.e., $n_- = 1$) to the length, respectively. Unlike the DLA perimeter case (because of the screening with the dead sites), all sites on the accessible perimeter have nonzero growth probability. It can be again shown that $l_n'' \sim 2[p_+' - p_-']n$. This result is checked in figure 3, according to which we obtain that the length of the accessible perimeter l_n'' grows linearly with the cluster size n as $\frac{l_n''}{n} = 1.203(2)$. It can thus be inferred that the incident square-like random walker attaches to the active zone of the cluster with a probability of $p_+' \gtrsim 60\%$ on choosing the single-side contact. Since all fjords of narrow throat are filled with dead sites inside the accessible perimeter, we can estimate the number of sites $\mathcal{N}_-$ on it with triple-side contact possibility by closing off all the narrow passageways of a lattice spacing. This yields the outer part of the accessible perimeter whose length l_n''' also has a linear relationship with the size of the cluster, i.e., as shown in figure 3, $\frac{l_n'''}{n} = 1.068(2)$. Comparing with the same relation for l_n'', one can conclude that $\mathcal{N}_- \sim 0.067n$. However, on closing off the narrow passageways, all possible remaining contacts will be single-side or double-side ones; nevertheless it is not possible to estimate an exact value for p_+. This is because these attachments can also make a negative contribution to the length of the external accessible perimeter.

These linear relationships seem to be characteristic features of DLA clusters. In fact, for common fractals appearing in two-dimensional statistical mechanics, such as critical Ising or percolation clusters, the length of the cluster boundaries (or loops) l with fractal dimension d_{f} has a scaling relation with the area of the loops, as $A \sim l^{2/d_{\mathrm{f}}}$.

Motivated by this relation, we examine the behavior of the length of the perimeters versus the area enclosed by them. As depicted in figure 4, we find that all the perimeter lengths l_n', l_n'' and l_n''' have linear relations with their area. Note that this area

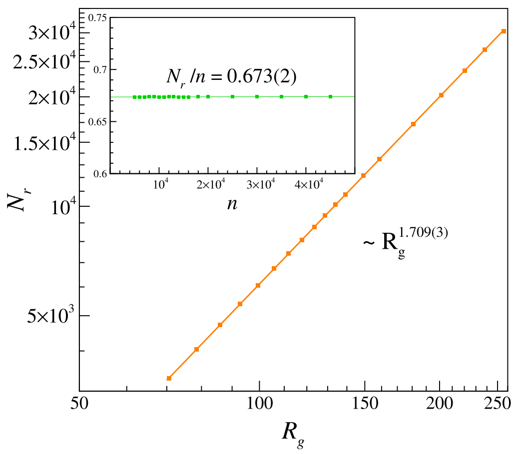

Figure 5. Main frame: the average number of red sites N_r versus the gyration radius R_g of DLA clusters of different sizes. Inset: linear relation between N_r and the cluster size n.

is not necessarily equal to the cluster size (multiplied by the square lattice spacing a^2). These linear relationships show that applying the scaling relation $A \sim l^{2/d_f}$ to measure the fractal dimension of the perimeter of DLA clusters would thus lead to a misleading result, as reported in [10].

The fractal dimension of the perimeters d_f can be computed by using the scaling relation between the average length of the perimeter l and a linear size scale, e.g., the gyration radius R_g, i.e., $l \sim R_g^{d_f}$. As long as the gyration radius of the growing cluster is used as the linear size scale in this relation, we find that the fractal dimensions of all perimeter lengths, within the statistical errors, are equal to the same value: the fractal dimension of the DLA cluster. This result is in agreement with the same one reported in [11]. Nevertheless, if we scale the perimeter lengths with their gyration radius (i.e., the gyration radius of the loops produced by the perimeters themselves), we find minor differences among the fractal dimensions. If d_f^c, d_f, d_f', d_f'' and d_f''' represent the fractal dimensions of the DLA cluster, the perimeter, the external perimeter, the accessible perimeter and the accessible external perimeter, respectively, we find that $d_f^c = 1.707(3)$, $d_f = 1.710(3)$, $d_f' = 1.717(3)$, $d_f'' = 1.723(3)$ and $d_f''' = 1.725(3)$.

In the following, we use the concept of *red sites* borrowed from two-dimensional critical structures [12] to have a quantitative measure for the ramification of the DLA cluster. A red site on the DLA cluster is a site such that cutting it leads to a splitting of the cluster. This measures the number of nodes connected by effectively one-dimensional links.

We carried out simulations in order to compute the fractal dimension of the red sites d_r on the DLA cluster by using the scaling relation $N_r \sim R_g^{d_r}$, where N_r is the number of red sites and R_g is the gyration radius of the DLA cluster. Due to the large amount of time needed, the simulation was performed for clusters of size $n \leqslant 4.5 \times 10^4$. As shown in figure 5, we find that, within statistical errors, the fractal dimension of the red sites is equal to that of the DLA cluster, i.e., $d_r = 1.709(3)$.

We also find that the number of red sites grows linearly with the cluster size according to the relation $\frac{N_r}{n} = 0.673(2)$.

In conclusion, we found that different surface parameters for DLA clusters such as the length of the hull, the accessible perimeter and their external parts, and also the number of dead sites, grow linearly with the cluster size. These findings have been used to investigate the microscopic features of the cluster growth by measuring the probability that a square-like incident random walker attaches to the cluster by choosing single-side, double-side or triple-side contacts. We also found that the border of the DLA grows linearly with the total area enclosed by it. We have measured the fractal dimension of the red sites on the DLA cluster as equal to that of the cluster itself. The average number of red sites has been shown to have a linear relation with the cluster size.

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